Two New Tests for Tail Independence in Extreme Value Models

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This paper proposes two new tests for tail independence in extreme value models. We use the conditional distribution function (df) of $X + Y$, given that $X + Y > c$ based approach of Falk and Michel to test for tail independence in extreme value models. We recommend using Cramer-von Mises and Anderson-Darling tests for tail independence. Simulations show that the two tests are better than the Kolmogorov-Smirnov test which has good results among the proposed tests by Falk and Michel. Finally, by using two real datasets, we illustrate the application of the two proposed tests as well as the traditional tests of Falk and Michel.

Keywords: *extreme value model, tail independence, Copula function, Cramer-von Mises test, Anderson-Darling test, Neyman-Pearson test, Kolmogorov-Smirnov test, Fisher's κ test, Chi-square goodness-of-fit test*

1. Introduction

Tail dependence describes the amount of dependence in the tail of a bivariate distribution. In other words, tail dependence refers to the degree of dependence in the corner of the lower-left quadrant or upper-right quadrant of a bivariate distribution. Definitions of tail dependence for multivariate random vectors are mostly related to their bivariate marginal df's. Geffroy (1958, 1959) and Sibuya (1960) independently introduced the quantity

$$\lambda_u := \lim_{t \to 1^-} P\left(X > F_X^{-1}(t) \mid Y > F_Y^{-1}(t) \right),$$

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where F_X^{-1} and $F_Y^{-1}\mathbb{F}$ are quasi-inverses of F_X and F_Y respectively. This quantity is called the upper tail dependence coefficient provided the limit exists, which is displayed for simplicity as TDC. We say that (X, Y) has upper tail dependence if $\lambda_u > 0$ and upper tail independent or asymptotically independent if $\lambda_u = 0$. Loosely speaking, tail dependence describes the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. Several empirical surveys such as An'e and Kharoubi (2003) and Malevergne and Sornette (2004) exhibited that the concept of tail dependence is a useful tool to describe the dependence between extremal data. The TDC can also be defined via the notion of copula. The copula function $C(u,v)$ is a bivariate df with uniform marginals on [0,1], such that $F(x,y) = C(F_X(x), F_Y(y))$. By Sklar's Theorem (Sklar, 1959), this copula exists and is unique if F_X and F_Y are continuous. Also, the copula C is given by $C(u,v) = F(F_X^{-1}(u), F_Y^{-1}(v)),\ \forall u,v \in [0,1]$ (for more details, see Nelsen, 2006). If $C(u,v)$ is the copula of (X, Y), then

$$\lambda_u = \lim_{t \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}.$$

See Coles et al. (1999). Frahm et al. (2005) introduced estimators for TDC under various assumptions: using a specific distribution, within a class of distributions, using a specific copula function, and within a class of copulas or a nonparametric estimation (without any parametric assumption).

In this paper we restrict our attention to extreme value copulas, i.e., a copula C such that

$$C(u,v) = \exp\left\{ log(uv) A\left(\frac{log(v)}{log(uv)} \right) \right\}, \quad u,v \in [0,1]^2, \tag{1}$$

where, A:[0,1] $\to$ [1/2,1] is the Pickands dependence function (Pickands 1981). This function is absolutely continuous and convex, satisfies $A(0) = A(1) = 1$, and its derivative has values between -1 and 1. When $A(t) = 1$, Equation (1) yields independence and when in Equation (1) we choose $A(t) = \max\{t, 1-t\}$, then complete dependence obtain. These copulas are useful to model componentwise maxima.

Let (X,Y) be a random vector (rv) with values in $(-\infty,0)^2$, whose df $H(x, y)$ coincides, for $x, y \leq 0$ close to 0, with a max-stable or extreme value df (EV) G with reverse exponential margins, i.e.,

$$G(x, 0) = G(0, x) = exp(x), \quad x \leq 0, \tag{2}$$

and

$$G^n\left(\frac{x}{n},\frac{y}{n}\right)=G(x,y),\quad x,y\le 0,n\in\mathbb{N}.$$

Suppose that $(X_1,Y_1),\ldots,(X_n,Y_n)$ are independent copies of (X,Y). If diagnostic checks of $(X_1,Y_1),\ldots,(X_n,Y_n)$ suggest X,Y to be independent in their upper tail, then modeling with dependencies leads to the over estimation of probabilities of extreme joint events. Some inference problems caused by model mis-specification are, for example, discussed in Dupuis and Tawn (2001). Testing for tail independence is, therefore, mandatory in a data analysis of extreme values.

Falk and Michel (2006) showed that the conditional df of $X+Y$, given that $X+Y>c$, has a limiting df $F(t)=t^2$, $t\in[0,1]$, as c $\uparrow$ 0 if and only if X, Y are tail independent. Otherwise, the limiting df is uniform distribution on [0,1], i.e., $F(t)=t$, $t\in[0,1]$. This result will be utilized to define tests for the tail independence of X, Y which are suggested by the Neyman-Pearson lemma as well as via the goodness-of-fit tests that are based on Fisher's κ, on the Kolmogorov-Smirnov test as well as on the chi-square goodness-of-fit test, applied to the exceedances $X_i+Y_i>c$ in the sample $(X_1,Y_1),\ldots,(X_n,Y_n)$. Using this approach we recommend Cramer-von Mises and Anderson-Darling tests for tail independence.

The organization of the paper is as follows. The next section briefly presents the approach of Falk and Michel (2006) and then expresses their tests for tail independence in extreme value models. Also, we introduce the two proposed tests based on the Cramer-von Mises and Anderson-Darling statistics. Section 3 compares the size and power of the proposed tests as well as the traditional tests for tail independence using Monte Carlo experiments. In Section 4, all tests mentioned in Section 2, are implemented on two real datasets. Finally, conclusions are given in the last section. In this paper, for computation and simulation, we use the R statistical software.

2. Tail Independence Tests

In the following, we assume that the rv (X,Y) has a df $H(x,y)$, which coincides, for $x,y\le 0$ close to 0, with a max-stable or extreme value df (EV) G with reverse exponential margins (Equation (2)). The following theorem from Falk and Michel (2006) is the basis of the tail independence tests in this paper.

Theorem 1. We have uniformly for $t\in[0,1]$ as c $\uparrow$ 0 as

$$P(X+Y>ct\mid X+Y>c)=\begin{cases}t^2(1+O(c)), \text{Tail Independence},\\ t(1+O(c)), \textit{elsewhere}.\end{cases}$$

Based on this theorem, Falk and Michel (2006) introduced four tests for tail independence in extreme value models, which can be grouped into two different classes: one based on Neyman-Pearson lemma and the other tests based on Fisher's κ, Kolmogorov-Smirnov and chi-square goodness-of-fit tests. These tests are presented below.

2.1. Proposed tests by Falk and Michel

Suppose that $(X_1,Y_1),\ldots,(X_n,Y_n)$ are independent copies of (X,Y). Fix $c < 0$ and consider only those observations X_i,Y_i among the sample that satisfy $X_i + Y_i > c$. Denote these by $C_1, C_2,\ldots, C_{K(n)}$ in the order of their outcome. If c is large enough, then C_i / c, $i = 1, 2,\ldots$ are iid with a common df F_C and are independent of $K(n)$, which is binomial $B(n, q)$ distributed with $q = 1-(1-c)\exp(c)$.

Neyman-Pearson Test. The first test Falk and Michel (2006) introduced is based on Neyman-Pearson lemma. We have to decide, roughly, whether the df of $V_i := C_i / c$, $i = 1, 2,\ldots$ is equal to either the null hypothesis $F_{(0)}(t) = t^2$ or the alternative $F_{(1)}(t) = t$, $0 \leq t \leq 1$. Assuming that these approximations of the df of $V_i := C_i / c$ are exact and that $K(n) = m > 0$, the optimal test for testing $F_{(0)}$ against $F_{(1)}$ is based on the loglikelihood ratio

$$T_{NP} := log\left(\prod_{i=1}^{m}\frac{1}{2V_i}\right) = -\sum_{i=1}^{m} log(V_i) - m\, log(2),$$

if m is large enough, the p-value of this test obtained by using the central limit theorem, that is equal to

$$p_{NP} = \Phi\left(\frac{2\sum_{i=1}^{m} log(V_i) + m}{m^{1/2}}\right),$$

where Φ denotes the df of the standard normal distribution.

The other three tests of Falk and Michel (2006) are goodness-of-fit tests based on C_i / c.

Fisher's κ Test. Conditioning on $K(n) = m > 0$, we consider the rvs

$$U_i := F_c(C_i / c) = \frac{1-(1-C_i)\exp(C_i)}{1-(1-c)\exp(c)}, \quad i = 1,\ldots,m,$$

if X and Y are tail independent and c is close to 0, according to Theorem 1, rvs U_i $(i=1,\ldots,m)$ are iid from uniform distribution on (0,1). Consider the corresponding order statistics $U_{1:m} \leq \ldots \leq U_{m:m}$ and define

$$S_j := U_{j:m} - U_{j-1:m}, \quad j = 2,\ldots,m,$$

and let $S_1 = U_{1:m}$, $S_{m+1} = 1 - U_{m:m}$. Suppose that

$$M_m := \max_{1 \le j \le m+1} S_j,$$

then, the Fisher's κ test statistic is

$$\kappa_m := (m+1)\, M_m.$$

A table of the critical values of Fisher's κ test is given in Fuller (1976). The p-value of this test is equal to

$$p_\kappa := 1 - G_{m+1}\left(\frac{\kappa_m}{m+1}\right) = 1 - G_{m+1}(M_m),$$

where

$$G_{m+1}(x) = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (\max(0, 1 - jx))^m, \quad x > 0.$$

Kolmogorov-Smirnov Test. Conditioning on $K(n) = m > 0$, we can apply the Kolmogorov-Smirnov test to rvs U_i $(i = 1, \ldots, m)$. Denote $\hat{F}_m(t) := \frac{1}{m} \sum_{i=1}^{m} I_{[0,t]}(U_i)$ be the empirical df of rvs U_i $(i = 1, \ldots, m)$, then the Kolmogorov-Smirnov statistic is

$$T_{KS} := m^{1/2} \sup_{t \in [0,1]} |\hat{F}_m(t) - t|.$$

The approximate p-value of Kolmogorov-Smirnov test is equal to

$$p_{KS} := 1 - K(T_{KS}),$$

where K is the df of the Kolmogorov distribution.

Chi-square Test. Conditioning on $K(n) = m > 0$, we can apply the chi-square goodness-of-fit test to rvs U_i $(i = 1, \ldots, m)$. For this purpose, we divide the interval [0,1] into k consecutive and disjoint intervals $I_1, \ldots, I_k$ and consider the chi-square statistic

$$\chi^2_{m,k} := \sum_{i=1}^{k} \frac{(m_i - mp_i)^2}{mp_i},$$

where m_i is the number of observations among $U_1, \ldots, U_m$ that fall into the interval I_i and p_i is the length of $I_i, 1 \le I \le k$. If m is large, such that for all $i = 1, \ldots, k$ we have $mp_i > 5$, then the statistic $\chi^2_{m,k}$ have chi-square distribution with k–1 degrees of freedom. Therefore, the approximate p-value of this test is equal to

$$p_{\chi^2} := 1 - \chi^2_{k-1}(\chi^2_{m,k}).$$

2.2. The proposed tests

Based on Theorem 1 from Falk and Michel (2006) we propose two new tests for tail independence in extreme value models. These tests are based on Cramer-von Mises and Anderson-Darling statistics.

Cramer-von Mises Test. Conditioning on $K(n) = m > 0$, we can apply the Cramer-von Mises test to rvs U_i ($i = 1, \ldots, m$). Consider the corresponding order statistics $U_{1:m} \leq \ldots \leq U_{m:m}$, then the Cramer-von Mises statistic is

$$T_{CM} := \frac{1}{12m} + \sum_{i=1}^{m} \left[U_{i:m} - \frac{2i-1}{2m} \right]^2.$$

Csorgo and Faraway (1996) obtained the exact and asymptotic dfs of Cramer-von Mises statistic, where we can use them to calculate p-value of this test. Therefore, approximate p-value of Cramer-von Mises test is equal to

$$p_{CM} := 1 - K(T_{CM}),$$

where K is the df proposed by Csorgo and Faraway (1996).

Anderson-Darling Test. Conditioning on $K(n) = m > 0$, we can apply the Anderson-Darling test to rvs U_i ($i = 1, \ldots, m$). Consider the corresponding order statistics $U_{1:m} \leq \ldots \leq U_{m:m}$, then the Anderson-Darling statistic is

$$T_{AD} := -m - \frac{1}{m} \sum_{i=1}^{m} (2i-1)[\log(U_{i:m}) + \log(1 - U_{m-i+1:m})].$$

Anderson and Darling (1954) found the limiting df of this statistic. The mean of this limiting df is 1 and the variance is $2(\pi^2 - 9)/3 \sim 0.57974$. Using the limiting df, we can obtain approximate p-value of Anderson-Darling test as below

$$p_{AD} := 1 - A(T_{AD}),$$

where A is the limiting df proposed by Anderson and Darling (1954).

3. Monte Carlo Experiments

In this section, we carried out to evaluate the performance of all above tests for the tail independence by using Monte Carlo experiments. The joint behavior of rv (X,Y) is assumed to be adequately represented by three one-parameter families of extreme value copulas with dependence parameter θ, namely Gumbel copula, Galambos copula and Husler-Reiss copula. Also, we considered Frank copula does not belong to extreme value copulas. The Gumbel copula is defined as

$$C_\theta(u,v) = exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}}\right\}, \quad \theta \in [1,\infty),$$

Galambos copula is expressed as

$$C_\theta(u,v) = uv\; exp\left\{-\left[(-\ln u)^{-\theta} + (-\ln v)^{-\theta}\right]^{-\frac{1}{\theta}}\right\}, \quad \theta \in [0,\infty),$$

for $\theta \in [0, \infty)$ Husler-Reiss copula is

$$C_\theta(u,v) = exp\left\{\ln u\; \Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\left(\frac{\ln u}{\ln v}\right)\right) + \ln v\; \Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\left(\frac{\ln v}{\ln u}\right)\right)\right\},$$

and Frank copula is specified by

$$C_\theta(u,v) = -\frac{1}{\theta} log\left[1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right], \quad \theta \in (-\infty,\infty)\setminus\{0\}.$$

For more details about these copulas see Joe (2014).

The Monte Carlo experiments are conducted for the threshold $c = -0.5, -0.1, -0.05$, and based on $K(n) = m = 25$ exceedances under the hypothesis H_0 of the independence of X and Y.

The chi-square statistic uses $k = 4$ intervals of equal length. 10000 replications are performed and we compute the percentage of rejection of H_0. Two characteristics of the tests were of interest: their ability to maintain their nominal level, arbitrarily fixed at 5% throughout the study, and their power under a variety of alternatives. It should be noted that, conditioning on $K(n) = m = 25$, when the threshold c increases to zero, the required sample size increases too.

Tables 1-3 give the percentage of rejection of the hypothesis of the independent tails of X and Y in sampling from different extreme value copulas. In Gumbel, Galambos and Husler-Reiss copulas, the TDC are equal to $2 - 2^{1/\theta}$, $2^{-1/\theta}$ and $2[1-\Phi(1/\theta)]$ respectively. Therefore, in each table, the first row of each test shows the empirical size of the test under the null hypothesis of the tail independence of rv (X,Y) and other rows present the power of these tests under the tail dependence.

Table 1. Percentage of rejection of H_0 by various tests with the underlying Gumbel copula with degrees of dependence θ and 25 exceedances over the threshold c

Test	Dependence Parameter θ	Threshold		
		-0.5	-0.1	-0.05
Neyman-Pearson	1	**0.1550**	**0.0797**	**0.0672**
	2	0.9704	0.9641	0.9703
	5	0.9852	0.9726	0.9698
	10	0.9843	0.9740	0.9701
Fisher's κ	1	**0.0500**	**0.0531**	**0.0494**
	2	0.1991	0.2388	0.2450
	5	0.2290	0.2405	0.2501
	10	0.2299	0.2486	0.2494
Kolmogorov-Smirnov	1	**0.0467**	**0.0515**	**0.0521**
	2	0.6236	0.7267	0.7513
	5	0.7140	0.7485	0.7586
	10	0.7222	0.7542	0.7604
Chi-square	1	**0.0365**	**0.0423**	**0.0407**
	2	0.4720	0.5841	0.6066
	5	0.5682	0.6050	0.6161
	10	0.5750	0.6077	0.6060
Cramer-von Mises	1	**0.0477**	**0.0492**	**0.0536**
	2	0.6841	0.7839	0.8050
	5	0.7702	0.8050	0.8112
	10	0.7742	0.8042	0.8072
Anderson-Darling	1	**0.0468**	**0.0490**	**0.0537**
	2	0.7960	0.8694	0.8879
	5	0.8622	0.8858	0.8893
	10	0.8647	0.8898	0.8913

As seen in tables regardless of the threshold value, except for the Neyman-Pearson test, the size of all tests is close to nominal level 5%, this is shown Bold in Tables 1-3. Of course, by choosing the small threshold close to 0 we ensure that the size of the Neyman-Pearson test also controls. This is inspected in Lemma 3.1 of Falk and Michel (2006).

Table 2. Percentage of rejection of H_0 by various tests with the underlying Galambos copula with degrees of dependence θ and 25 exceedances over the threshold c

Test	Dependence Parameter θ	Threshold		
		-0.5	-0.1	-0.05
Neyman-Pearson	0	**0.1688**	**0.0906**	**0.0917**
	2	0.9805	0.9674	0.9713
	5	0.9856	0.9713	0.9708
	10	0.9853	0.9729	0.9721
Fisher's κ	0	**0.0485**	**0.0528**	**0.0523**
	2	0.2104	0.2351	0.2424
	5	0.2304	0.2415	0.2460
	10	0.2335	0.2386	0.2415
Kolmogorov-Smirnov	0	**0.0510**	**0.0500**	**0.0498**
	2	0.6742	0.7392	0.7571
	5	0.7132	0.7453	0.7535
	10	0.7165	0.7509	0.7557
Chi-square	0	**0.0434**	**0.0400**	**0.0368**
	2	0.5266	0.5938	0.6083
	5	0.5758	0.6064	0.6130
	10	0.5671	0.6100	0.6119
Cramer-von Mises	0	**0.0536**	**0.0502**	**0.0523**
	2	0.7282	0.7918	0.8058
	5	0.7698	0.8013	0.8068
	10	0.7698	0.8106	0.8063
Anderson-Darling	0	**0.0550**	**0.0527**	**0.0545**
	2	0.8306	0.8771	0.8896
	5	0.8616	0.8878	0.8886
	10	0.8622	0.8873	0.8872

Comparison of the power of the tests shows that the Neyman-Pearson test having the largest power followed by the Anderson-Darling, Cramer-von Mises, Kolmogorov-Smirnov and chi-square tests, respectively.

Table 3. Percentage of rejection of H_0 by various tests with the underlying Husler-Reiss copula with degrees of dependence θ and 25 exceedances over the threshold c

Test	Dependence Parameter θ	Threshold		
		-0.5	-0.1	-0.05
Neyman-Pearson	0	**0.1652**	**0.0737**	**0.0633**
	2	0.9774	0.9716	0.9705
	5	0.9847	0.9700	0.9701
	10	0.9870	0.9723	0.9684
Fisher's κ	0	**0.0487**	**0.0507**	**0.0497**
	2	0.1974	0.2348	0.2509
	5	0.2251	0.2485	0.2496
	10	0.2288	0.2438	0.2421
Kolmogorov-Smirnov	0	**0.0484**	**0.0497**	**0.0522**
	2	0.6602	0.7382	0.7509
	5	0.7118	0.7464	0.7556
	10	0.7245	0.7398	0.7577
Chi-square	0	**0.0373**	**0.0389**	**0.0391**
	2	0.5111	0.5895	0.6047
	5	0.5603	0.6049	0.6119
	10	0.5810	0.5994	0.6121
Cramer-von Mises	0	**0.0526**	**0.0485**	**0.0532**
	2	0.7186	0.7886	0.8013
	5	0.7641	0.8000	0.8067
	10	0.7801	0.7984	0.8155
Anderson-Darling	0	**0.0512**	**0.0496**	**0.0524**
	2	0.8234	0.8774	0.8846
	5	0.8599	0.8832	0.8850
	10	0.8684	0.8811	0.8885

As Falk and Michel (2006) pointed out the distribution of p_κ is almost not affected, therefore the test for the independence of X and Y based on Fisher's κ fails. These results are viewable in Tables 1-3.

Table 4. Percentage of rejection of H_0 by various tests with the underlying Frank copula with degrees of dependence θ and 25 exceedances over the threshold c

Test	Dependence Parameter θ	Threshold		
		-0.5	-0.1	-0.05
Neyman-Pearson	0	0.1589	0.0739	**0.0621**
	2	0.3928	0.1094	**0.0804**
	5	0.6683	0.1626	**0.1053**
	10	0.8722	0.2726	**0.1564**
Fisher's κ	0	0.0502	0.0479	**0.0471**
	2	0.0655	0.0547	**0.0512**
	5	0.1021	0.0548	**0.0552**
	10	0.1550	0.0703	**0.0525**
Kolmogorov-Smirnov	0	0.0502	0.0494	**0.0447**
	2	0.0997	0.0572	**0.0491**
	5	0.2434	0.0737	**0.0557**
	10	0.4726	0.1161	**0.0729**
Chi-square	0	0.0403	0.0424	**0.0374**
	2	0.0664	0.0437	**0.0372**
	5	0.1592	0.0519	**0.0433**
	10	0.3329	0.0760	**0.0518**
Cramer-von Mises	0	0.0521	0.0491	**0.0439**
	2	0.1086	0.0577	**0.0507**
	5	0.2829	0.0763	**0.0584**
	10	0.5252	0.1322	**0.0786**
Anderson-Darling	0	0.0505	0.0507	**0.0458**
	2	0.1161	0.0604	**0.0499**
	5	0.3050	0.0783	**0.0577**
	10	0.5660	0.1389	**0.0806**

Table 4 illustrates the percentage of rejection of the hypothesis of the independent tails of X and Y in sampling from Frank copula. In Frank copula, for all values of the dependence parameter θ, TDC is equal to zero; i.e. X and Y are tail independent. Therefore, this table shows the empirical size of the test under the null hypothesis of the tail independence of rv (X,Y). As seen in Table 4, when the dependence parameter θ is zero (i.e. data does not have any dependency), except for the Neyman-Pearson test, the size of all tests is close to nominal level 5% and by choosing the small threshold the size of the Neyman-Pearson test also controls. By increasing the dependence parameter, although X and Y do not have tail dependence, the empirical size of the tests are violated. Looking at Table

4, we observe that in this case if the threshold value is close to 0, the empirical level approaches the nominal level, this is shown Bold in Table 4. The results of Table 4 show that, even if rv (X,Y) does not belong to extreme value model, tail independence tests for a small threshold still have good performance.

4. Data Analysis

In this section, the application of tail independence tests is illustrated using two different datasets. The first one is due to Cornwell and Trumbull (1994), who prepared based on the transcript of crime in North Carolina regarding 24 variables. The dataset included a panel of 90 observational units (counties) from 1981 to 1987, i.e. total number of observations is 630. We consider the two variables density (people per square mile) and crmrte (crimes committed per person) and other variables are ignored. We consider this dataset as Crime data. The second dataset, reported from "Investing.com." This site is a global financial portal and internet brand composed of 28 editions in 21 languages and mobile apps for Android and iOS that provide news, analysis, streaming quotes and charts, technical data and financial tools about the global financial markets. We consider stock price pairs from two Japanese multinational automaker: Honda Motor and Mazda Motor. Our sample period covers a total 758 observations from 10 Sep. 2014 to 16 Oct. 2017. We call this dataset as Stock data. In Figure 1, we draw scatter plots of empirical df of pairs for two datasets.

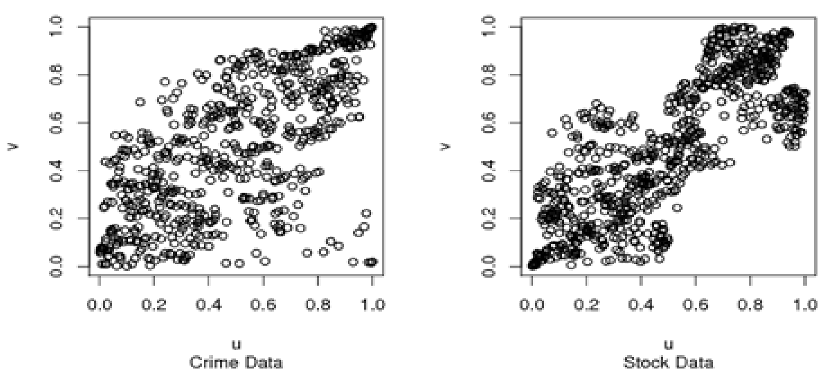

Figure 1. Scatter Plots of Empirical df of Pairs

We use a specific copula method for estimating TDC. For this purpose, we fitted three famous Archimedean copulas to the two datasets and obtained Cramer-von Mises statistic $S_n^{(B)}$ introduced by Genest et al. (2009), where is based on Rosenblatt's transform. It should be noted that the margins are estimated by empirical dfs. The results are shown in Table 5.

Table 5. Copula goodness-of-fit test for two datasets

Copula under H_0	Crime Data				Stock Data			
	p.value	AIC	$\hat{\theta}$	TDC	p.value	AIC	$\hat{\theta}$	TDC
Clayton	0	-313.88	1.909	-----	0	-556.56	2.620	-----
Frank	0.097	-368.61	5.528	-----	0.093	-644.07	**7.112**	**0**
Gumbel	**0.24**	**-375.09**	**1.954**	**0.574**	0.032	-580.55	2.310	-----

According to the p-values of tests, we conclude that Gumbel copula and Frank copula have best fit to the two datasets respectively. Therefore Crime data are tail dependent, where TDC is equal to 0.574 and Stock data are tail independent. In the following, all proposed tests in Section 2 are performed on the two datasets and the results are displayed in Table 6. It should be noted that in carrying out these tests, for each dataset, the threshold c is chosen to have at least 30 observations greater than of the threshold value. Therefore, in two datasets, the thresholds are equal to –0.15 and –0.25 respectively.

Table 6. Independence tests for two datasets

Test	p.value	
	Crime Data	Stock Data
Neyman-Pearson	4.891685e-09	0.7543855
Fisher's κ	4.364887e-02	0.2194695
Kolmogorov-Smirnov	1.245545e-03	0.4993588
Chi-square	1.514254e-02	0.6754989
Cramer-von Mises	1.027966e-03	0.7278006
Anderson-Darling	3.082995e-04	0.6549564

In Crime data, all tests reject the null hypothesis of the tail independence of variables density and crmrte at 0.05 level, i.e., two variables density and crmrte are tail dependent; therefore, if the density of people per square mile exceeds a certain threshold, then crimes committed per person will exceed that specific threshold.

In Stock data, tail independence is not rejected by any of the tests at 0.05 level, i.e., stock prices of the two Japanese automakers Honda and Mazda are tail independent. Therefore tail independence tests confirmed the results of Table 5. It is noteworthy that if the TDC is estimated using the unsuitable copula function, the tail independence tests show this matter; this indicates the importance of using the test to verify the existence of tail dependence in the data.

5. Conclusion

In this paper, we recommended two new statistics Cramer-von Mises and Anderson-Darling for tail independence in extreme value models-based approach of Falk and Michel (2006). Simulations show that two tests are better than the proposed tests by Falk and Michel. Also, we illustrated the importance of using these tests by using two real datasets, while the tail dependence maybe is estimated incorrectly and this wrong is shown by tests.

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