On Some Efficient Classes of Estimators Based on Higher Order Moments of an Auxiliary Attribute

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Abstract

This paper discusses the problem of estimating the population mean utilizing information on the mean and variance of qualitative characteristics. We introduce some efficient classes of estimators based on higher order moments such as the variance of an auxiliary attribute. The conventional mean estimator, Bhushan and Gupta (2016) estimator, and the traditional regression and ratio estimators proposed by Naik and Gupta (1996) are shown to be the sub-class of the proposed estimators for properly chosen valuations of the described scalars. The effective performance of the suggested estimators has been assessed empirically and theoretically with respect to the contemporary estimators.

Keywords: *mean square error, efficiency, qualitative characteristics*

1. Introduction

In survey research, it has been found that the consideration of auxiliary (supplementary) information assists to improve the efficiency of the estimator provided that there arises a high degree of correlation between the variable of interest and the supplementary variable. Several classes of improved and modified estimators have been suggested till date. Bhushan et al. (2020a, b, c, 2021a, b), Bhushan and Kumar (2020, 2022), and the references listed therein are a few recent noteworthy contributions in this regard. In real life scenarios, many times the study variable may be associated with some easily available qualitative auxiliary elements. For example: height of persons (y) and sex (ϕ); quantity of production of gram crop (y) and a specific variety of gram crop (ϕ); quantity of milk production (y) and a specific breed of cow (ϕ). Moreover, if measuring a quantitative variable is costly, then such an auxiliary attribute may be considered, which can be constructed from the auxiliary variable and is highly associated with the study variable. For example: (1) the tax paid by a company (y) may depend on its turnover (ϕ) which can be converted into a large/small companies; (2) the family expenditure (y) may depend on the household size (ϕ) which can be classified as large/small household; (3) the yield of crop (y) may depend on large/small land holdings (ϕ).

Various renowned authors suggested a wide range of classes of modified and improved population mean estimators consisting of auxiliary attributes in simple random sampling (SRS). The classical product, regression and ratio estimators were suggested by Naik and Gupta (1996) for the population mean estimation utilizing auxiliary attribute. Jhajj et al.

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(2006) investigated a general class of population mean estimator based on auxiliary attribute. Influenced by the work of Ray and Singh (1981), Singh et al. (2008) developed a class of estimators utilizing auxiliary attribute. Adapting the procedure of Kadilar and Cingi (2006), Abd-Elfattah et al. (2010) developed a class of population mean estimators by combining different ratio estimators. Grover and Kaur (2011) extended their own work and investigated an improved population mean exponential estimator utilizing auxiliary attribute. Singh and Solanki (2012) proposed a more effective auxiliary attribute-based estimation method for population mean. Koyuncu (2012) envisaged an efficient population mean estimator utilizing auxiliary attribute. Bhushan and Gupta (2016) introduced a ratio type estimators based on higher order moments such as the variance of an attribute ϕ. A family of ratio exponential estimators consisting of an auxiliary attribute were explored by Zaman and Kadilar (2019). Zaman (2020) developed an exponential kind of estimators utilizing attribute. An improved auxiliary attribute-based log type estimator was suggested by Bhushan and Gupta (2020). Motivated by Bhushan et al. (2021b), Bhushan et al. (2022a) suggested a few attribute-based enhanced classes of estimators for population mean. In order to compute the population mean and variance utilizing auxiliary attribute, Bhushan et al. (2022b) presented certain linear combination type estimators.

This paper discusses some efficient auxiliary attribute-based classes of regression and ratio type estimators using higher order moments such as the variance. The following sections make up the schema of this paper. *Section* 2 devotes to the existing population mean estimators based on auxiliary attribute. In *Section* 3, a few efficient classes of regression and ratio type estimators have been proposed using higher order moments such as variance of auxiliary attribute along with their characteristics. The comparative study between the suggested and existing estimators has been performed in *Section* 4. Results of an empirical application are presented in *Section* 5 for the verification of the efficiency conditions followed by the discussion of the numerical results tabulated in Section 6. Finally, we reach to the conclusion in *Section* 7.

2. Review of Existing Estimators

To compute the population mean $\bar{Y}$ of the study variable y, let a simple random sample s of size n be drawn without replacement from a finite population $\aleph = (\aleph_1, \aleph_2, \ldots, \aleph_N)$ of size *N*. Let ϕ_i and y_i be the observations on the auxiliary attribute ϕ and the variable of choice y for unit i of the population $\aleph$. Note that if the unit $i \in \phi$ then ϕ_i=1 and if $i \notin \phi$ then ϕ_i=0. Suppose $A = \Sigma_{i=1}^{N}\phi_i$ and $a = \Sigma_{i=1}^{n}\phi_i$ represent the total number of units in the population $\aleph$ and sample s, respectively, with attribute ϕ, whereas $P = (A/N)$ and $p = (a/n)$, respectively, represent the population and sample proportion with attribute ϕ. Let $\bar{Y} = N^{-1}\Sigma_{i=1}^{N}y_i$ and $\bar{y} = n^{-1}\Sigma_{i=1}^{n}y_i$ be, respectively the population and sample means of study variable, y with the expressions $S_y = \sqrt{(N-1)^{-1}\Sigma_{I=1}^{N}(y_i - \bar{Y})^2}$ and $S_\phi = \sqrt{(N-1)^{-1}\Sigma_{I=1}^{N}(\phi_i - P)^2}$ be, respectively, the population standard deviation of the study variable y and the auxiliary attribute ϕ. Furthermore, let us define $S_{y\phi} = (N-1)^{-1}(\Sigma_{i=1}^{N}y_i\phi_i - N\bar{Y}P)$ and $\rho = S_{y\phi}/S_yS_\phi$ be the population covariance and population coefficient of correlation between the study variable y and the auxiliary attribute ϕ. Likewise, $C_y = S_y/\bar{Y}$ and $C_\phi = S_\phi/P$ be, respectively, the coefficient of variation of study variable y and auxiliary attribute ϕ.

We take into consideration the following expressions to define the mean square error (MSE) and bias of the estimators: $\bar{y}=\bar{Y}\,(1+e_0)$, $p=P(1+e_1)$ and $s_\phi^2=S_\phi^2(1+e_2)$ given $E(e_0)=E(e_1)=E(e_2)=0$ and $E(e_0^2)=\gamma C_y^2$, $E(e_1^2)=\gamma C_\phi^2$, $E(e_2^2)=\gamma(\lambda_{04}-1)$, $E(e_0e_1)=\gamma\rho C_yC_\phi$ $E(e_0e_2)=\gamma C_y\lambda_{12}$ and $E(e_1e_2)=\gamma\lambda_{03}C_\phi$ where $\gamma=(N-n)/Nn$, $\lambda_{ab}=\mu_{ab}/\mu_{20}^{a/2}\mu_{02}^{b/2}$, and $\mu_{ab}=(N-1)^{-1}\Sigma_{i=1}^{N}(y_i-\bar{Y})^a(\phi_i-P)^b$.

The classical mean estimator is stated as $T_m=\bar{y}$ while the variance of the estimator T_m is expressed as

$$V(T_m)=\gamma\bar{Y}^2C_y^2\,. \tag{1}$$

An attribute-based classical ratio estimator was presented by Naik and Gupta (1996) $T_r=\bar{y}\left(\frac{P}{p}\right)$ while the MSE of the estimator T_r is given by

$$MSE(T_r)=\gamma\bar{Y}^2\left(C_y^2+C_\phi^2-2\rho C_yC_\phi\right). \tag{2}$$

An attribute-based classical regression estimator was also presented by Naik and Gupta (1996) as

$$T_{lr}=\bar{y}+\beta_\phi(P-p),$$

where β_ϕ is the regression coefficient of y on ϕ. The MSE of the estimator T_{lr} is provided as

$$MSE(T_{lr})=\gamma\bar{Y}^2\left(C_y^2+\beta_\phi^2\frac{P^2}{\bar{Y}^2}C_\phi^2-2\beta_\phi\frac{P}{\bar{Y}}\rho C_yC_\phi\right).$$

By minimizing the $MSE(T_{lr})$ with respect to (w.r.t.) β_ϕ, we obtain $\beta_{\phi(opt)}=\rho\left(\frac{\bar{Y}C_y}{PC_\phi}\right)$. By replacing the value of β_ϕ with $\beta_{\phi(opt)}$ in the $MSE(T_{lr})$, we obtain

$$minMSE(T_{lr})=\bar{Y}^2\gamma C_y^2(1-\rho^2). \tag{3}$$

Motivated by the work of Ray and Singh (1981), Singh et al. (2008) investigated the undermentioned estimators utilizing information on auxiliary attribute as

$$T_s=\left\{\bar{y}+\beta_\phi(P-p)\right\}\left(\frac{m_1P+m_2}{m_1p+m_2}\right),$$

where β_ϕ is the same as defined earlier, $m_1(\neq 0)$ and m_2 are either real numbers or functions of the known parameters of the attribute, namely, the population coefficient of kurtosis $\beta_2(\phi)$, the population coefficient of variation C_ϕ of the attribute ϕ and the population correlation coefficient ρ between the variable y and the attribute ϕ. Furthermore, some members of the estimator T_s are given hereunder for the suitably chosen values of constants m_1 and m_2 as

$$T_{s_1}=\left\{\bar{y}+\beta_\phi(P-p)\right\}\left(\frac{P}{p}\right),$$

$$T_{s_2}=\left\{\bar{y}+\beta_\phi(P-p)\right\}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right),$$

$$T_{s_3} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P+C_\phi}{p+C_\phi}\right),$$

$$T_{s_4} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P\beta_2(\phi)+C_\phi}{p\beta_2(\phi)+C_\phi}\right),$$

$$T_{s_5} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{PC_\phi+\beta_2(\phi)}{pC_\phi+\beta_2(\phi)}\right),$$

$$T_{s_6} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P+\rho}{p+\rho}\right),$$

$$T_{s_7} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{PC_\phi+\rho}{pC_\phi+\rho}\right),$$

$$T_{s_8} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P\rho+C_\phi}{p\rho+C_\phi}\right),$$

$$T_{s_9} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P\beta_2(\phi)+\rho}{p\beta_2(\phi)+\rho}\right),$$

$$T_{s_{10}} = \{\bar{y} + \beta_\phi(P - p)\}\left(\frac{P\rho+\beta_2(\phi)}{p\rho+\beta_2(\phi)}\right),$$

where $\beta_2(\phi) = \mu_{04}/\mu_{02}^2$ is the population coefficient of kurtosis of attribute ϕ. The MSE of the estimator $T_{s_i}, i = 1, 2, \ldots, 10$ is provided by

$$MSE(T_{s_i}) = \gamma\bar{Y}^2\{R_i^2C_\phi^2 + C_y^2(1 - \rho^2)\}, \tag{4}$$

Where $R_1 = 1$, $R_2 = P/(P + \beta_2(\phi))$, $R_3 = P/(P + C_\phi)$, $R_4 = P\beta_2(\phi)/(P\beta_2(\phi) + C_\phi)$, $R_5 = PC_\phi/(PC_\phi + \beta_2(\phi))$, $R_6 = P/(P + \rho)$, $R_7 = PC_\phi/(PC_\phi + \rho)$, $R_8 = P\rho/(P\rho + C_\phi)$, $R_9 = P\beta_2(\phi)/(P\beta_2(\phi) + \rho)$, and $R_{10} = P\rho/(P\rho + \beta_2(\phi))$.

Bhushan et al. (2015) suggested the auxiliary attribute-based logarithmic estimator under SRS as $T_{bg} = \bar{y}\left\{1 + \alpha\log\left(\frac{p^*}{P^*}\right)\right\}$, where α is a duly selected scalar. Additionally, $p^* = \eta p + \lambda$ and $P^* = \eta P + \lambda$ such that η and λ are either real numbers or functions of the known population parameters of the attribute, namely, the population coefficient of kurtosis $\beta_2(\phi)$, the population coefficient of variation C_ϕ of auxiliary attribute and the population correlation coefficient ρ between y and ϕ. The MSE of the estimator T_{bg} is given by

$$MSE(T_{bg}) = \gamma\bar{Y}^2(C_y^2 + \alpha^2\nu^2C_\phi^2 + 2\alpha\nu\rho C_yC_\phi),$$

where $\nu = \eta P/(\eta P + \lambda)$. By minimizing the $MSE(T_{bg})$ w.r.t. α, we get $\alpha_{(opt)} = -\rho\left(\frac{C_y}{\nu C_\phi}\right)$

and by replacing the value of α with $\alpha_{(opt)}$ in the $MSE(T_{bg})$, we get

$$minMSE(T_{bg}) = \bar{Y}^2\gamma C_y^2(1 - \rho^2). \tag{5}$$

This resembles the minimum MSE of the estimator for classical regression.

Bhushan and Gupta (2016) introduced the following class of estimator utilizing higher order moments such as the variance of an auxiliary attribute as $T_{bg1} = \bar{y}\left(\frac{P}{p}\right)^{\beta}\left(\frac{S_{\phi}^2}{s_{\phi}^2}\right)^{\theta}$, where β and θ are scalars. The MSE of the estimator T_{bg1} is given by

$$MSE(T_{bg1}) = \gamma\bar{Y}^2\{C_y^2 + \beta^2 C_{\phi}^2 + \theta^2(\lambda_{04} - 1) - 2\beta\rho C_y C_{\phi} + 2\beta\theta C_{\phi}\lambda_{03} - 2\theta C_y \lambda_{12}\}.$$

By minimizing the $MSE(T_{bg1})$ with respect to β and θ, we get $\beta_{(opt)} = \left(\frac{S_y}{S_{\phi}}\right)\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}$, and $\theta_{(opt)} = \left(\frac{S_y}{S_{\phi}^2}\right)\frac{\rho(\lambda_{12}-\rho\lambda_{03})}{(\lambda_{04}-1-\lambda_{03}^2)}$. By replacing the value of β and θ with $\beta_{(opt)}$, and $\theta_{(opt)}$, respectively, in the $MSE(T_{bg1})$, we get

$$minMSE(T_{bg1}) = \gamma\bar{Y}^2 C_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12} - \rho\lambda_{03})^2}{(\lambda_{04} - 1 - \lambda_{03}^2)}\right\}. \tag{6}$$

We would like to mention that following Bhushan and Gupta (2016), one may also propose regression and Walsh (1970) type estimators using higher order moments such as the variance of auxiliary attribute but both will furnish similar results under the optimal conditions.

Following Singh et al. (2007), Zaman and Kadilar (2019) suggested the following estimator as $T_{zk} = \bar{y}\exp\left\{\frac{(\eta P+\lambda)-(\eta p+\lambda)}{(\eta P+\lambda)+(\eta p+\lambda)}\right\}$, where η and λ are same as previously specified. The MSE of the estimator T_{zk} is expressed as

$$MSE(T_{zk}) = \gamma\bar{Y}^2(\zeta^2 C_{\phi}^2 + C_y^2 - 2\zeta\rho C_y C_{\phi}), \tag{7}$$

where $\zeta = \eta P/2(\eta P + \lambda)$. A few members of the estimator T_{zk} are also included in Table 1 for quick reference. Following Ozel (2016), Zaman (2020) introduced an exponential ratio kind of estimator as $T_z = \bar{y}\left(\frac{p}{P}\right)^{\varsigma}\exp\left\{\frac{(\eta P+\lambda)-(\eta p+\lambda)}{(\eta P+\lambda)+(\eta p+\lambda)}\right\}$, where ς is a suitably chosen scalar and η and λ are same as previously specified. The MSE of the estimator T_z is expressed by

$$MSE(T_z) = \gamma\bar{Y}^2(C_y^2 + \varsigma^2 C_{\phi}^2 + \nu^2 C_{\phi}^2 - 2\varsigma\nu C_{\phi}^2 + 2\varsigma\rho C_y C_{\phi} - 2\nu\rho C_y C_{\phi}),$$

where $\nu = \eta P/(\eta P + \lambda)$. By minimizing the $MSE(T_z)$ w.r.t. ς, we get $\varsigma_{(opt)} = \nu - \frac{\rho C_y}{C_{\phi}}$. By replacing the value of ς with $\varsigma_{(opt)}$ in the $MSE(T_z)$, we get

$$minMSE(T_z) = \bar{Y}^2\gamma C_y^2(1 - \rho^2), \tag{8}$$

which is similar to the minimum MSE of the classical regression estimator T_{lr}.

Bhushan and Gupta (2020) suggested the log type family of estimator utilizing auxiliary attribute as $T_{bg2} = \left\{w_1\bar{y} + w_2\left(\frac{p}{P}\right)\right\}\left\{1 + \alpha\log\left(\frac{p^*}{P^*}\right)\right\}$, where w_1, w_2 and α are duly chosen scalars. Moreover, the optimum value of α can be determined from the log kind of estimators T_{bg} envisaged by Bhushan et al. (2015). Furthermore, p^* and P^* are the same as defined earlier. A few members of the estimator T_{bg2} are also included in Table 1 for quick reference.

Table 1: Some members of Zaman and Kadilar (2019) and Bhushan and Gupta (2020) estimators.

Values of $\boldsymbol{\eta}$	Values of λ	Members of Zaman and Kadilar (2019) estimators $T_{zk(j)}, j = 1, 2, \ldots, 10$	Members of Bhushan and Gupta (2020) estimator $T_{bg2(j)}, j = 1,2, \ldots, 10$
1	$\beta_2(\phi)$	$T_{zk(1)}$	$T_{bg2(1)}$
1	C_ϕ	$T_{zk(2)}$	$T_{bg2(2)}$
$\boldsymbol{\beta_2(\phi)}$	C_ϕ	$T_{zk(3)}$	$T_{bg2(3)}$
$\boldsymbol{C_\phi}$	$\beta_2(\phi)$	$T_{zk(4)}$	$T_{bg2(4)}$
1	ρ	$T_{zk(5)}$	$T_{bg2(5)}$
$\boldsymbol{C_\phi}$	ρ	$T_{zk(6)}$	$T_{bg2(6)}$
$\boldsymbol{\rho}$	C_ϕ	$T_{zk(7)}$	$T_{bg2(7)}$
$\boldsymbol{\beta_2(\phi)}$	ρ	$T_{zk(8)}$	$T_{bg2(8)}$
$\boldsymbol{\rho}$	$\beta_2(\phi)$	$T_{zk(9)}$	$T_{bg2(9)}$
$\boldsymbol{S_\phi}$	$\beta_2(\phi)$	$T_{zk(10)}$	$T_{bg2(10)}$

The MSE of the estimator T_{bg2} is given by

$$MSE(T_{bg2}) = (\bar{Y}^2 w_1^2 A + w_2^2 B + \bar{Y}^2 w_1 D + \bar{Y} w_2 G + \bar{Y} w_1 w_2 F + \bar{Y}^2), \tag{9}$$

where $A = 1 + \gamma(C_y^2 + \alpha^2 v^2 C_\phi^2 + 4\alpha v \rho C_y C_\phi - \alpha v C_\phi^2)$, $B = 1 + \gamma(C_\phi^2 + \alpha^2 v^2 C_\phi^2 - \alpha v \rho C_\phi^2 + 4\alpha v C_\phi^2)$, $D = \gamma(\alpha v^2 C_\phi^2 - 2\alpha v \rho C_y C_\phi) - 2$, $G = \gamma(\alpha v^2 C_\phi^2 - 2\alpha v C_\phi^2) - 2$ and $F = 2 + 2\gamma(2\alpha v C_\phi^2 + 2\alpha v \rho C_y C_\phi + \rho C_y C_\phi - \alpha v^2 C_\phi^2 + \alpha^2 v^2 C_\phi^2)$.

By minimizing the $MSE(T_{bg2})$ w.r.t. w_1 and w_2, we obtain $w_{1(opt)} = \frac{(GF-2BD)}{(4AB-F^2)}$, and $w_{2(opt)} = \frac{\bar{Y}(DF-2GA)}{(4AB-F^2)}$. By replacing the values of w_1 and w_2 with $w_{1(opt)}$ and $w_{2(opt)}$, respectively, in the $MSE(T_{bg2})$, we obtain

$$minMSE(T_{bg2}) = \bar{Y}^2 \left\{1 - \frac{(AG^2 + BD^2 - 2DFG)}{(4AB - F^2)}\right\}. \tag{10}$$

3. Proposed Classes of Estimators

The main goal of this article is to use data on the auxiliary characteristics to develop some efficient classes of estimators as an alternative to the estimators reviewed in the preceding section. Motivated by the work of Searls (1964) and Bhushan and Gupta (2016), we suggest some regression and ratio type estimators using higher order moments such as the variance of an auxiliary attribute ϕ as

$$T_{a1} = \alpha_1 \bar{y} + \beta_1 (P - p) + \theta_1 (S_\phi^2 - s_\phi^2),$$

$$T_{a2} = \alpha_2 \bar{y} \left(\frac{P}{p}\right)^{\beta_2} \left(\frac{S_\phi^2}{s_\phi^2}\right)^{\theta_2}, \text{ and}$$

$$T_{a3} = \alpha_3 \bar{y} \left\{\frac{P}{P+\beta_3(p-P)}\right\} \left\{\frac{S_\phi^2}{S_\phi^2+\theta_3\left(s_\phi^2-S_\phi^2\right)}\right\},$$

where α_i, β_i and θ_i, $i = 1, 2, 3$ are suitably opted scalars. The classes of estimators T_{ai}, $i = 1, 2, 3$ are reduced into:

- Usual mean estimator T_m for $(\alpha_i, \beta_i, \theta_i) = (1,0,0)$,
- Classical regression estimator T_{lr} for $(\alpha_1, \beta_1, \theta_1) = \left(1, \beta_\phi, 0\right)$,
- Classical ratio estimator T_r for $(\alpha_2, \beta_2, \theta_2) = (1, 1, 0)$
- Bhushan and Gupta (2016) estimator T_{bg1} for $(\alpha_2, \beta_2, \theta_2) = (1, \beta, \theta)$.

Theorem 3.1. The suggested class of estimator T_{a1} has the following bias and minimum MSE:

$$Bias(T_{a1}) = \bar{Y}(\alpha_1 - 1), \tag{11}$$

$$minMSE(T_{a1}) = \bar{Y}^2\left(1 - \alpha_{1(opt)}\right). \tag{12}$$

Proof. Consider the first estimator

$$T_{a1} = \alpha_1 \bar{y} + \beta_1 (P - p) + \theta_1\left(S_\phi^2 - s_\phi^2\right).$$

Rewrite the estimator T_{a1} by utilizing the notations defined in Section 2, we get

$$T_{a1} = \alpha_1 \bar{Y} + \alpha_1 \bar{Y} e_0 - \beta_1 P e_1 - \theta_1 S_\phi^2 e_2.$$

By subtracting $\bar{Y}$ on both sides of the above expression, we get

$$T_{a1} - \bar{Y} = \alpha_1 \bar{Y} + \alpha_1 \bar{Y} e_0 - \beta_1 P e_1 - \theta_1 S_\phi^2 e_2 - \bar{Y}. \tag{13}$$

The bias of the estimator T_{a1} is obtained by taking the expectation on both sides of *Equation* 13 which results to $Bias(T_{a1}) = \bar{Y}(\alpha_1 - 1)$. On the other hand, the MSE of the estimator T_{a1} is obtained by squaring and taking the expectation on both sides of *Equation* 13 as

$$MSE(T_{a1}) = \left\{(\alpha_1 - 1)^2 \bar{Y}^2 + \alpha_1^2 \gamma S_y^2 + \beta_1^2 \gamma S_\phi^2 + \theta_1^2 \gamma S_\phi^4 (\lambda_{04} - 1) - 2\alpha_1 \beta_1 \gamma \rho S_y S_\phi - 2\alpha_1 \theta_1 \gamma S_y S_\phi^2 \lambda_{12} + 2\beta_1 \theta_1 \gamma S_\phi^3 \lambda_{03}\right\}.$$

By minimizing the $MSE(T_{a1})$ w.r.t. α_1, β_1 and θ_1, we get

$$\alpha_{1(opt)} = \frac{\bar{Y}^2}{\bar{Y}^2 + \gamma S_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12} - \rho\lambda_{03})^2}{(\lambda_{04} - 1 - \lambda_{03}^2)}\right\}},$$

$$\beta_{1(opt)} = \alpha_{1(opt)}\left(\frac{S_y}{S_\phi}\right)\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\},$$

$$\theta_{1(opt)} = \alpha_{1(opt)}\left(\frac{S_y}{S_\phi^2}\right)\frac{(\lambda_{12}-\rho\lambda_{03})}{(\lambda_{04}-1-\lambda_{03}^2)}.$$

By substituting the optimum values of $\alpha_1,\ \beta_1$ and θ_1 in the $MSE(T_{a1})$, we get

$$\begin{aligned} minMSE(T_{a1}) = \Bigg[&\left(\alpha_{1(opt)} - 1\right)^2\bar{Y}^2 + \alpha_{1(opt)}^2\gamma S_y^2 + \alpha_{1(opt)}^2\left(\frac{S_y}{S_\phi}\right)^2\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}^2\gamma S_\phi^2 + \\ &\alpha_{1(opt)}^2\left(\frac{S_y}{S_\phi^2}\right)^2\left\{\frac{\lambda_{12}-\rho\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}^2\gamma S_\phi^4(\lambda_{04}-1) - \\ &2\alpha_{1(opt)}^2\left(\frac{S_y}{S_\phi}\right)\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}\gamma\rho S_y S_\phi - \\ &2\alpha_{1(opt)}^2\left(\frac{S_y}{S_\phi^2}\right)\frac{(\lambda_{12}-\rho\lambda_{03})}{(\lambda_{04}-1-\lambda_{03}^2)}\gamma S_y S_\phi^2\lambda_{12} + \\ &2\alpha_{1(opt)}^2\left(\frac{S_y^2}{S_\phi^3}\right)\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}\left\{\frac{\lambda_{12}-\rho\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}\gamma S_\phi^3\lambda_{03}\Bigg]. \end{aligned}$$

After simplification of the above equation, we get

$$\begin{aligned} minMSE(T_{a1}) &= \left[\left(\alpha_{1(opt)}^2 + 1 - 2\alpha_{1(opt)}\right)\bar{Y}^2 + \alpha_{1(opt)}^2\gamma S_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12}-\rho\lambda_{03})^2}{(\lambda_{04}-1-\lambda_{03}^2)}\right\}\right], \\ &= \left(1 - 2\alpha_{1(opt)}\right)\bar{Y}^2 + \alpha_{1(opt)}^2\left[\bar{Y}^2 + \gamma S_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12}-\rho\lambda_{03})^2}{(\lambda_{04}-1-\lambda_{03}^2)}\right\}\right], \\ &= \left(1 - 2\alpha_{1(opt)}\right)\bar{Y}^2 + \alpha_{1(opt)}^2\left(\frac{\bar{Y}^2}{\alpha_{1(opt)}}\right), \\ &= \bar{Y}^2\left(1 - \alpha_{1(opt)}\right). \end{aligned}$$

Theorem 3.2. The suggested classes of estimator $T_{ai}, i = 2,\ 3$ have the following bias and minimum MSE:

$$Bias(T_{ai}) = \bar{Y}(\alpha_i Q_i - 1), \tag{14}$$

$$\min MSE(T_{ai}) = \bar{Y}^2\left(1 - \frac{Q_i^2}{P_i}\right). \tag{15}$$

Proof. Consider the estimator T_{a2} as

$$T_{a2} = \alpha_2 \bar{y} \left(\frac{P}{p}\right)^{\beta_2} \left(\frac{S_\phi^2}{s_\phi^2}\right)^{\theta_2}.$$

Rewriting the above estimator by utilizing the notation defined in *Section* 2, we get

$$T_{a2} = \alpha_2 \bar{Y}(1+e_0)(1+e_1)^{-\beta_2}(1+e_2)^{-\theta_2},$$
$$= \alpha_2 \bar{Y}(1+e_0)\left\{1 - \beta_2 e_1 + \tfrac{\beta_2(\beta_2+1)}{2} e_1^2 + \cdots\right\}\left\{1 - \theta_2 e_2 + \tfrac{\theta_2(\theta_2+1)}{2} e_2^2 + \cdots\right\}.$$

Using Taylor series expansion, we expand the right-hand side of the above expression, multiplying out and neglecting the terms of e's having power greater than two, we get

$$T_{a2} = \alpha_2 \bar{Y}\left\{1 + e_0 - \beta_2 e_1 - \beta_2 e_0 e_1 + \tfrac{\beta_2(\beta_2+1)}{2} e_1^2 - \theta_2 e_2 - \theta_2 e_0 e_2 + \tfrac{\theta_2(\theta_2+1)}{2} e_2^2\right\}.$$

By subtracting $\bar{Y}$ on both sides of the above expression, we get

$$T_{a2} - \bar{Y} = \bar{Y}\left[\alpha_2\left\{1 + e_0 - \beta_2 e_1 - \beta_2 e_0 e_1 + \frac{\beta_2(\beta_2+1)}{2} e_1^2 - \theta_2 e_2 - \theta_2 e_0 e_2 + \frac{\theta_2(\theta_2+1)}{2} e_2^2\right\} - 1\right].$$

Letting the above equation as *Equation* 16 and taking the expectation on both sides of the equation, we get the bias of the estimator T_{a2} up to the first order of approximation as

$$Bias(T_{a2}) = \bar{Y}\left[\alpha_2\left\{1 + \frac{\beta_2(\beta_2+1)}{2}\gamma C_\phi^2 + \frac{\theta_2(\theta_2+1)}{2}\gamma(\lambda_{04}-1) - \beta_2\gamma\rho C_y C_\phi - \theta_2\gamma C_y \lambda_{12}\right\} - 1\right]$$
$$= \bar{Y}(\alpha_2 Q_2 - 1).$$

The bias of the estimator T_{a3} can be obtained in the same manner.

Now, squaring and applying the expectation on both sides of *Equation* 16, we get the MSE up to the first order of approximation as

$$MSE(T_{a2}) = \bar{Y}^2\Big[1 + \alpha_2^2\{1 + \gamma C_y^2 + (2\beta_2^2 + \beta_2)\gamma C_\phi^2 + (2\theta_2^2 + \theta_2)\gamma(\lambda_{04}-1) - 4\beta_2\gamma\rho C_y C_\phi - 4\theta_2\gamma C_y \lambda_{12} + 4\beta_2\theta_2\gamma C_\phi \lambda_{03}\} - 2\alpha_2\left\{1 + \tfrac{\beta_2(\beta_2+1)}{2}\gamma C_\phi^2 + \tfrac{\theta_2(\theta_2+1)}{2}\gamma(\lambda_{04}-1) - \beta_2\gamma\rho C_y C_\phi - \theta_2\gamma C_y\lambda_{12} + \beta_2\theta_2\gamma C_\phi\lambda_{03}\right\}\Big],$$

which can further be written as

$$MSE(T_{a2}) = \bar{Y}^2(1 + \alpha_2^2 P_2 - 2\alpha_2 Q_2). \tag{17}$$

By minimizing the $MSE(T_{a2})$ w.r.t. α_2, we get $\alpha_{2(opt)} = \frac{Q_2}{P_2}$ and by replacing the value of α_2 with $\alpha_{2(opt)}$ in *Equation* 17, we get $minMSE(T_{a2}) = \bar{Y}^2\left(1 - \frac{Q_2^2}{P_2}\right)$. The MSE of the estimator T_{a3} can be determined on the same manner. In general, we can write

$$Bias(T_{ai}) = \bar{Y}(\alpha_i Q_i - 1);\ i = 2, 3,$$
$$MSE(T_{ai}) = \bar{Y}^2(1 + \alpha_i^2 P_i - 2\alpha_i Q_i). \tag{18}$$

By minimizing the $MSE(T_{ai})$ w.r.t. α_i, we get $\alpha_{i(opt)} = \frac{Q_i}{P_i}$ and by replacing the value of α_i with $\alpha_{i(opt)}$ in *Equation* 18 we get $\min MSE(T_{ai}) = \bar{Y}^2\left(1 - \frac{Q_i^2}{P_i}\right)$ where

$P_2 = 1 + \gamma\{C_y^2 + (2\beta_2^2 + \beta_2)C_\phi^2 + (2\theta_2^2 + \theta_2)(\lambda_{04} - 1) - 4\beta_2\rho C_y C_\phi - 4\theta_2 C_y\lambda_{12} + 4\beta_2\theta_2 C_\phi\lambda_{03}\}$,

$Q_2 = 1 + \gamma\left\{\frac{\beta_2(\beta_2+1)}{2}C_\phi^2 + \frac{\theta_2(\theta_2+1)}{2}(\lambda_{04} - 1) - \beta_2\rho C_y C_\phi - \theta_2 C_y\lambda_{12} + \beta_2\theta_2 C_\phi\lambda_{03}\right\}$,

$P_3 = 1 + \gamma\{C_y^2 + 3\beta_3^2 C_\phi^2 + 3\theta_3^2(\lambda_{04} - 1) - 4\beta_3\rho C_\phi C_y - 4\theta_3 C_y\lambda_{12} + 4\beta_3\theta_3 C_\phi\lambda_{03}\}$, and

$Q_3 = 1 + \gamma\{\beta_3^2 C_\phi^2 + \theta_3^2(\lambda_{04} - 1) - \beta_3\rho C_\phi C_y - \theta_3 C_y\lambda_{12} + \beta_3\theta_3 C_\phi\lambda_{03}\}$.

Furthermore, $\beta_{i(opt)} = \frac{S_y}{S_\phi}\left\{\frac{\rho(\lambda_{04}-1)-\lambda_{12}\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right\}$ and $\theta_{i(opt)} = \frac{S_y}{S_\phi^2}\left(\frac{\lambda_{12}-\rho\lambda_{03}}{\lambda_{04}-1-\lambda_{03}^2}\right), i = 2,3$ are utilized as the optimum values of scalars when $\alpha_i = 1$ is put in the corresponding estimators.

Corollary 3.1. The proposed classes of ratio type estimators $T_{ai}, i = 2,3$ are superior than the proposed regression type of estimator T_{a1} if and only if $\frac{Q_i^2}{P_i} > \alpha_{1(opt)}$.

Proof: This can be shown by the comparison of *Equations* 12 and 15.

We would also like to note that *Theorem* 3.1 and *Theorem* 3.2 are important to obtain the efficiency conditions discussed in *Section* 4.

4. Comparative Study

We deduce the efficiency conditions by comparing the minimum MSEs of the suggested classes of estimators and existing estimators.

- From *Equation* 12 and *Equation* 1, we get
  $MSE((T_m) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma C_y^2$.
- From *Equation* 15 and *Equation* 1, we get
  $MSE(T_m) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma C_y^2, i = 2,3$.
- From *Equation* 12 and *Equation* 2, we get
  $MSE(T_r) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma(C_y^2 + C_\phi^2 - 2\rho C_y C_\phi)$.
- From *Equation* 15 and *Equation* 2, we get
  $MSE(T_r) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma(C_y^2 + C_\phi^2 - 2\rho C_y C_{phi}), i = 2,3$.
- From *Equation* 12 and *Equation* 3, we get
  $MSE(T_{lr}) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma C_y^2(1 - \rho^2)$.
- From *Equation* 15 and *Equation* 3, we get
  $MSE(T_{lr}) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma C_y^2(1 - \rho^2), i = 2,3$.
- From *Equation* 12 and *Equation* 4, we get
  $MSE(T_{sj}) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma\{R_j^2 C_\phi^2 + C_y^2(1 - \rho^2)\}, j = 1,2,\ldots,10$.
- From *Equation* 15 and *Equation* 4, we get
  $MSE(T_{sj}) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma\{R_j^2 C_\phi^2 + C_y^2(1 - \rho^2)\}, i = 2,3$.
- From *Equation* 12 and *Equation* 6, we get
  $MSE(T_{bg1}) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma C_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12}-\rho\lambda_{03})^2}{\lambda_{04}-1-\lambda_{03}}\right\}$.
- From *Equation* 15 and *Equation* 6 , we get
  $MSE(T_{bg1}) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma C_y^2\left\{1 - \rho^2 - \frac{(\lambda_{12}-\rho\lambda_{03})^2}{\lambda_{04}-1-\lambda_{03}}\right\}, i = 2,3$.
- From *Equation* 12 and *Equation* 7, we get
  $MSE(T_{zk}) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma(\zeta^2 C_\phi^2 + C_y^2 - 2\zeta\rho C_y C_\phi), i = 2,3$.

- From *Equation* 15 and *Equation* 7, we get
  $MSE(T_{zk}) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma\left(\zeta^2 C_\phi^2 + C_y^2 - 2\zeta\rho C_y C_\phi\right), i = 2, 3.$
- From *Equation* 12 and *Equation* 8, we get
  $MSE(T_z) > MSE(T_{a1})$ when $\alpha_{1(opt)} > 1 - \gamma C_y^2(1 - \rho^2)$.
- From *Equation* 15 and *Equation* 8, we get
  $MSE(T_z) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > 1 - \gamma C_y^2(1 - \rho^2), i = 2, 3.$
- From *Equation* 12 and *Equation* 9, we get
  $MSE\left(T_{bg2}\right) > MSE(T_{a1})$ when $\alpha_{1(opt)} > \frac{AG^2 + BD^2 - 2DFG}{(4AB - F^2)}$.
- From *Equation* 15 and *Equation* 9, we get
  $MSE\left(T_{bg2}\right) > MSE(T_{ai})$ when $\frac{Q_i^2}{P_i} > \frac{AG^2 + BD^2 - 2DFG}{(4AB - F^2)}, i = 2, 3.$

Under the conditions mentioned above, the proposed estimators $T_{ai},\ i = 1, 2, 3$ outperform the traditional mean estimator, traditional ratio and regression estimators, Singh et al. (2008) estimators, Bhushan et al. (2015) estimator, Bhushan and Gupta (2016) estimator, Zaman and Kadilar (2019) estimators, Zaman (2020) estimator and Bhushan and Gupta (2020) estimators. Successively, an empirical assessment was conducted using two different real populations to verify the above efficiency conditions.

5. Empirical Assessment

This empirical application used two real populations which are discussed below.

Population 1. (Origin: Sukhatme and Sukhatme (1970), pp. 256)

y: area (in acres) under the wheat crop inside the circles,

ϕ: a circle based on more than five villages,

$N = 89,\ n = 23,\ \bar{Y} = 1102,\ P = 0.124, C_y = 0.65,\ C_\phi = 2.678,$ and $\rho = 0.624$.

Population 2. (Origin: Singh and Chaudhary (1986), pp. 141)

y: area under lime (in acres),

ϕ: number of bearing lime trees (> 500).

N =22, n =12, $\bar{Y}$ =22.62091, P =0.5, C_y =1.4609, C_ϕ =1.0235, and ρ =0.6292.

We now compute the MSE and percent relative efficiency (PRE) for the above two populations. The PRE is calculated for the existing and proposed estimators T w.r.t. the traditional mean estimator T_m utilizing the expression: $PRE = \frac{MSE(T_m)}{MSE(T)} \times 100$. The outcomes of this empirical asessement for both populations are presented in Table 2 by MSE and PRE, demonstrating how the suggested classes of estimators outperform the existing estimators presented in *Section* 2.

Table 2: MSE and PRE of different estimators

Estimators	Population 1		Population 2	
	MSE	PRE	MSE	PRE
T_m	16559.39	100.00	41.36	100.00
T_r	212259.00	7.80	25.21	164.07
T_{lr}	10111.56	163.76	25.00	165.46
T_{s1}	290912.50	5.69	45.30	91.30
T_{s2}	10220.13	162.02	25.11	164.71
T_{s3}	10658.08	155.36	27.18	152.15
T_{s4}	23878.64	69.34	36.44	113.51
T_{s5}	10841.23	152.74	25.12	164.67
T_{s6}	17786.42	93.10	28.98	142.72
T_{s7}	43841.90	37.77	29.08	142.21
T_{s8}	10331.60	160.27	26.12	158.35
T_{s9}	94946.42	17.44	39.00	106.05
T_{s10}	10154.47	163.07	25.04	165.14
T_{bg1}	10342.32	160.11	12.11	341.42
$T_{zk(1)}$	15731.42	105.15	40.02	103.34
$T_{zk(2)}$	14798.41	111.78	35.92	115.13
$T_{zk(3)}$	10561.38	156.63	30.53	135.45
$T_{zk(4)}$	14551.86	113.68	39.99	103.42
$T_{zk(5)}$	11421.86	144.83	34.28	120.64
$T_{zk(6)}$	10240.09	161.55	34.20	120.92
$T_{zk(7)}$	15404.10	107.39	37.36	110.72
$T_{zk(8)}$	14403.59	114.85	29.72	139.16
$T_{zk(9)}$	16026.43	103.22	40.49	102.15
$T_{zk(10)}$	16263.84	101.71	40.64	101.76
T_z	10111.56	163.76	25.00	165.46
$T_{bg2(1)}$	14142.35	116.97	17.01	243.05
$T_{bg2(2)}$	15209.51	108.76	16.45	251.35
$T_{bg2(3)}$	20479.82	80.77	11.95	345.93
$T_{bg2(4)}$	15492.64	106.78	16.96	243.76
$T_{bg2(5)}$	19245.03	85.96	12.76	323.98
$T_{bg2(6)}$	21995.06	75.21	15.15	272.92
$T_{bg2(7)}$	14516.17	113.96	12.39	333.81
$T_{bg2(8)}$	21362.40	77.44	13.28	311.45
$T_{bg2(9)}$	13805.71	119.82	17.83	231.91
$T_{bg2(10)}$	13534.92	122.22	18.10	228.43
T_{a1}	**10028.06**	**165.13**	**11.85**	**348.88**
T_{a2}	**9883.88**	**167.53**	**11.83**	**349.51**
T_{a3}	**10030.91**	**165.08**	**11.88**	**348.06**

6. Results and Discussion

From the reported findings in *Table* 2, the proposed estimators $\boldsymbol{T_{ai}}$, *i*=1,2,3 outperform the following:

- The traditional mean per unit estimator T_m, classical regression and ratio estimators T_{lr} and T_r, Bhushan et al. (2015) estimator T_{bg}, Bhushan and Gupta (2016) estimators T_{bg1}, Zaman (2020) estimator T_z.
- The members $T_{s_i}, i = 1,2, \dots ,10$ of Singh et al. (2008) estimator T_s.
- The member $T_{zk(j)}, j = 1, 2, \dots , 10$ of Zaman and Kadilar (2019) estimators T_{zk}.
- The members $T_{bg2(j)}, j = 1, 2, \dots , 10$ of Bhushan and Gupta (2020) estimators T_{bg2}.

Furthermore, the proposed class of estimator T_{a2} is found to be the most efficient among the proposed classes of estimators $T_{ai}, i = 1, 2, 3$ for both populations by having the minimum MSE and maximum PRE.

7. Conclusion

This manuscript suggests some efficient classes of regression and ratio kind of estimators for the computation of population mean $\bar{Y}$ of the variable of interest *y* utilizing a higher order moments such as the variance of an auxiliary attribute ϕ. The conventional mean estimator T_m, ratio estimator T_r, regression estimator T_{lr} and Bhushan and Gupta (2016) estimator T_{bg_1} are found to be the members of the envisaged classes of estimators for duly opted valuations of the characterizing scalars. The bias and MSE expressions have been obtained and the efficiency conditions have been derived by comparing the MSE of the proposed classes of estimators with the MSE of the contemporary estimators.

Furthermore, an empirical assessment utilized two real populations to verify the credibility of the efficiency conditions. The numerical results have been found to be highly rewarding with minimum MSE and maximum PRE exhibiting superiority over the traditional mean estimator T_m, classical regression and ratio estimators T_{lr} and T_r, members $T_{s_i}, i = 1,2, \dots ,10$ of Singh et al. (2008) estimator T_s, Bhushan et al. (2015) estimator T_{bg}, Bhushan and Gupta (2016) estimator T_{bg_1}, members $T_{zk(j)}$; *j* =1,2,...,10 of Zaman and Kadilar (2019) estimators T_{zk}, Zaman (2020) estimator T_z and members $T_{bg_2(j)}$; *j* =1,2,...,10 of Bhushan and Gupta (2020) estimators T_{bg_2}. Moreover, it has been also seen from the numerical results that the ratio type estimator T_{a_2} performs superior among the proposed estimators. Thus, due to their uncontested performance, the proposed estimators are highly recommended to the surveyors for the estimation of population mean $\bar{Y}$ of variable of interest *y* such that the information is present as auxiliary attribute ϕ.

Literature Cited:

ABD-ELFATTAH, A.M., EL-SHERPIENY, E.A., MOHAMED, S.M. and ABDOU, O.F. 2010. "Improvement in Estimating the Population Mean in Simple Random Sampling Using Information on Auxiliary Attribute." *Applied Mathematics and Computation* 215: 4198-4202.

BHUSHAN, S., GUPTA, R. and PANDEY, S.K. 2015. "Improved Searl's type Logarithmic Estimators Using Auxiliary Information." *Statistical and Mathematical Sciences & their Applications,* Narosa Publishing House, New Delhi.

____________ and GUPTA, R. 2016. "Efficient Class of Estimators for Population Mean Using Attribute." *Recent Advances in Applied Statistics and its Applications* 225-229.

____________ and GUPTA, R. 2020. "An Improved Log-Type Family of Estimators Using Attribute." *Journal of Statistics and Management Systems 23*(3): 593-602.

____________ and KUMAR, A. 2020. "Log Type Estimators of Population Mean Under Ranked Set Sampling." *Predictive Analytics using Statistics and Big Data: Concepts and Modelling* 28: 47-74.

____________ and KUMAR, A. 2022. "Novel Log Type Class of Estimators Under Ranked Set Sampling." *Sankhya B*, 84: 421-447.

____________, GUPTA, R., SINGH, S. and KUMAR, A. 2020a. "A Modified Class of Log Type Estimators for Population Mean Using Auxiliary Information on Variable." *International Journal of Applied Engineering Research* 15(6): 612-627.

____________, GUPTA, R., SINGH, S. and KUMAR, A. 2020b. "Some Improved Classes of Estimators Using Auxiliary Information." *International Journal for Research in Applied Science & Engineering Technology* 8(VI): 1088-1098.

____________, GUPTA, R., SINGH, S. and KUMAR, A. 2020c. "A New Efficient Log-Type Class of Estimators Using Auxiliary Variable." *International Journal of Statistics and Systems* 15(1): 19-28.

____________, KUMAR, A., KUMAR, S. and SINGH, S. 2021a. "An Efficient Class of Estimators of Population Mean Under Simple Random Sampling." *The Philippine Statistician* 70(1): 33-47.

____________, KUMAR, A. and SINGH, S. 2021b. "Some Efficient Classes of Estimators Under Stratified Sampling." *Communications in Statistics - Theory and Methods* 1-30. [Online] DOI:10.1080/03610926.2021.1939052.

____________, KUMAR, A., TYAGI, D. and SINGH, S. 2022a. "On Some Improved Classes of Estimators Under Stratified Sampling Using Attribute." *Journal of Reliability and Statistical Studies* 15(1): 187-210.

____________, KUMAR, A., KUMAR, S. and SINGH, S. 2022b. "Some Modified Classes of Estimators for Population Variance Using Auxiliary Attribute." *Pakistan Journal of Statistics* 38(2): 235-252.

GROVER, L.K. and KAUR, P. 2011. "An Improved Exponential Estimator of Finite Population Mean in Simple Random Sampling Using an Auxiliary Attribute." *Applied Mathematics and Computation* 218(7): 3093-3099.

JHAJJ, H.S., SHARMA, M.K., GROVER, L.K. 2006. "A Family of Estimators of Population Mean Using Information on Auxiliary Attribute." *Pakistan Journal of Statistics* 22(1): 43-50.

KADILAR, C. and CINGI, H. 2006. "Improvement in Estimating the Population Mean in Simple Random Sampling." *Applied Mathematics Letters* 19: 75-79.

KOYUNCU N. 2012. "Efficient Estimators of Population Mean Using Auxiliary Attributes." *Applied Mathematics and Computation* 218(22): 10900-10905.

NAIK, V.D. and GUPTA, P.C 1996. "A Note on Estimation of Mean With Known Population Proportion of An Auxiliary Character." *Journal of Indian Society of Agricultural Statistics* 48(2): 151-158.

OZEL, K.G. 2016. "A New Exponential Type Estimator for the Population Mean in Simple Random Sampling." *Journal of Modern Applied Statistical Methods* 15(2): 207-214.

RAY, S.K. and SINGH, R.K. 1981. "Difference-Cum-Ratio Type Estimators." *Journal of Indian Statistical Association* 19: 147-151.

SEARLS, D.T. 1964. "The Utilization of A Known Coefficient of Variation in the Estimation Procedure." *Journal of American Statistical Association* 59: 1225-1226.

SINGH, D. and CHAUDHARY, F.S. 1986. "*Theory and Analysis of Sample Survey Designs."* New Age International Limited, New Delhi.

SINGH, R., CHAUHAN, P., SAWAN, N. and SMARANDACHE, F. 2008. "Ratio Estimators in Simple Random Sampling Using Information on Auxiliary Attribute." *Pakistan Journal of Statistics and Operation Research* IV(1): 47-53.

SINGH, H.P. and SOLANKI, R.S. 2012. "Improved Estimation of Population Mean in Simple Random Sampling Using Information on Auxiliary Attribute." *Applied Mathematics and Computation* 218(15): 7798-7812.

SUKHATME, P.V. and SUKHATME, B.V 1970. "*Sampling Theory of Surveys with Applications."* Iowa State University Press, Ames, U.S.A.

ZAMAN, T. and KADILAR, C. 2019. "Novel Family of Exponential Estimators Using Information of Auxiliary Attribute." *Journal of Statistics and Management Systems 22*(8): 1499-1509.

__________. 2020. "Generalized Exponential Estimators for the Finite Population Mean." *Statistics in Transition New Series* 21(1): 159-168.